Tutorial 2 (23 Sep)

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Q1) Define the Fejér kernel $\{F_n : [-\pi,\pi] \to \mathbb{R}\}_{n=1}^{\infty}$ by $F_n(x) = \frac{1}{n}(D_0(x) + \cdots + D_{n-1}(x))$,

where $D_\ell(x) = \sum_{k=-\ell}^{\ell} e^{ikx} = \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}$ is the Dirichlet kernel. (which differs from lecture note by a multiplicative constant 2π)

(a) Show that for $x \neq 0$, $F_n(x) = \frac{1}{n}\frac{\sin^2(\frac{nx}{2})}{\sin^2\frac{x}{2}}$.

(b) Show that $\{F_n\}_{n=1}^{\infty}$ is a good kernel, i.e satisfying the following properties:

1. $\forall n \in \mathbb{N}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)dx = 1$.

2. $\exists M > 0$ s.t. $\forall n \in \mathbb{N}$, $\int_{-\pi}^{\pi} |F_n(x)| dx \leq M$.

3. $\forall \delta > 0$, $\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} |F_n(x)| dx = 0$.

Sol) (a) Idea: Evaluate D_n and their sums via geometric progression formula.

Let $a = e^{ix} \neq 1$, recall that for each $\ell \in \mathbb{N}$,

$$D_\ell(x) = \sum_{k=-\ell}^{\ell} e^{ikx} = \sum_{k=-\ell}^{\ell} a^k = a^{-\ell}\left(\frac{1-a^{2\ell+1}}{1-a}\right) = \frac{a^{-\ell} - a^{\ell+1}}{1-a}.$$

$$\therefore \forall n \in \mathbb{N},\ n \cdot F_n(x) = \sum_{\ell=0}^{n-1} D_\ell(x) = \sum_{\ell=0}^{n-1}\left(\frac{a^{-\ell} - a^{\ell+1}}{1-a}\right) = \frac{1}{1-a}\left(\sum_{\ell=0}^{n-1} a^{-\ell} - \sum_{\ell=0}^{n-1} a^{\ell+1}\right)$$

$$= \frac{1}{1-a}\left[\left(\frac{1-a^{-n}}{1-a^{-1}}\right) - a\left(\frac{1-a^n}{1-a}\right)\right] = \frac{1}{(1-a)^2}\left(-a(1-a^{-n}) - a(1-a^n)\right) = \frac{a}{(1-a)^2}\left(a^{-n} - 2 + a^n\right)$$

$$= \frac{(a^{\frac{n}{2}} - a^{-\frac{n}{2}})^2}{(a^{\frac{1}{2}} - a^{-\frac{1}{2}})^2} = \frac{\left(2i\left(e^{\frac{inx}{2}} - e^{-\frac{inx}{2}}\right)\right)^2}{\left(2i\left(e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}\right)\right)^2} = \frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}.$$

$$\therefore F_n(x) = \frac{1}{n}\frac{\sin^2(\frac{nx}{2})}{\sin^2\frac{x}{2}}.$$

$\boxed{\ }$

(b) Idea: Use the formula in (a) to verify ②, ③.

①: $\forall n \in \mathbb{N}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{n}\sum_{\ell=0}^{n-1} D_\ell(x)\right)dx$

$$= \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} D_\ell(x)\,dx\right) = \frac{1}{n}\left(\sum_{\ell=0}^{n-1} 1\right) = 1.$$

②: $\forall n \in \mathbb{N}$, $\forall 0 \neq x \in [-\pi,\pi]$, $F_n(x) = \frac{1}{n}\frac{\sin^2(\frac{nx}{2})}{\sin^2\frac{x}{2}} \geq 0$; also, $F_n(0) = \frac{1}{n}\left(\sum_{\ell=0}^{n-1} 1\right) = 1$.

$\therefore \forall x \in [-\pi,\pi]$, $F_n(x) \geq 0$. Hence $\int_{-\pi}^{\pi}|F_n(x)|dx = \int_{-\pi}^{\pi} F_n(x)dx = 2\pi$ by ①.

$\therefore$ Choose any $M \geq 2\pi$, then $\forall n \in \mathbb{N}$, $\int_{-\pi}^{\pi}|F_n(x)|dx \leq M$.

③: $\forall \delta > 0$, showing $\sin^2\frac{x}{2} \geq \sin^2\frac{\delta}{2}$, $\forall \delta \leq |x| \leq \pi$:

$\forall \delta \leq x \leq \pi$, $\sin^2\frac{x}{2} \geq \sin^2\frac{\delta}{2}$; $\forall -\pi \leq x \leq -\delta$, $\sin\frac{x}{2} \leq \sin(-\frac{\delta}{2}) \leq 0 \Rightarrow \sin^2\frac{x}{2} \geq \sin^2(-\frac{\delta}{2}) = \sin^2\frac{\delta}{2}$.

$\therefore \forall \delta \leq |x| \leq \pi$, $\sin^2\frac{x}{2} \geq \sin^2\frac{\delta}{2}$. Hence

$$0 \leq \int_{\delta\leq|x|\leq\pi}|F_n(x)|dx = \int_{\delta\leq|x|\leq\pi}\frac{1}{n}\frac{\sin^2(\frac{nx}{2})}{\sin^2\frac{x}{2}}dx \leq \frac{1}{n}\int_{\delta\leq|x|\leq\pi}\frac{1}{\sin^2\frac{\delta}{2}}dx = \frac{1}{n}\cdot\frac{2(\pi-\delta)}{\sin^2\frac{\delta}{2}} \to 0 \text{ as } n\to\infty.$$

$$\therefore \lim_{n\to\infty}\int_{\delta\leq|x|\leq\pi}|F_n(x)|dx = 0.$$

Rmk • The Dirichlet kernel $\{D_n\}_{n=1}^{\infty}$ is NOT a good kernel

because ② does not hold: $\lim_{n\to\infty}\int_0^{\delta}|D_n(z)|dz = \infty$, $\forall \delta > 0$

• Meanwhile, its "average" $\{F_n = \frac{1}{n}\sum_{\ell=0} D_\ell\}_{n=1}^{\infty}$ is "better" in the sense of next question.

Q2) Let $f: \mathbb{R} \to \mathbb{C}$ be 2π-periodic continuous.

(a) Show that for any $x \in [-\pi, \pi]$, we have

$$\lim_{n\to\infty} (f * F_n)(x) := \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)F_n(y)\,dy = f(x)$$

where $F_n(x)$ is defined as in Q1.

(b) Hence, show that if in addition $c_k(f) = 0, \forall k \in \mathbb{Z}$, then

$f(x) = 0$ for any $x \in [-\pi, \pi]$.

Sol) (a) Idea: Following similar argument as in showing local convergence of Fourier series of Lipschitz continuous functions.

Note that $\forall n \in \mathbb{N}$, $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)F_n(y)\,dy\right) - f(x)$

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)F_n(y)\,dy\right) - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)F_n(y)\,dy\right) \quad \text{(By Q1b. ①)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) F_n(y)\,dy$$

$\therefore$ It suffices to show that $\lim_{n\to\infty} \int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) F_n(y)\,dy = 0.$

Given $\varepsilon > 0$, by continuity of f at x, there exists $\delta > 0$ such that for all $|y| \leq \delta$,

$$|f(x-y) - f(x)| < \varepsilon.$$

By Q1b. ③ applying to δ, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\int_{\delta \leq |y| \leq \pi} |F_n(y)|\, dy < \varepsilon.$$

$$\therefore \left| \int_{-\pi}^{\pi} (f(x-y)-f(x))\, F_n(y)\, dy \right| = \left| \int_{-\delta}^{\delta} (f(x-y)-f(x))\, F_n(y)\, dy + \int_{\delta \leq |y| \leq \pi} (f(x-y)-f(x))\, F_n(y)\, dy \right|$$

$$\leq \int_{-\delta}^{\delta} |f(x-y)-f(x)|\,|F_n(y)|\, dy + \int_{\delta \leq |y| \leq \pi} |f(x-y)-f(x)|\,|F_n(y)|\, dy$$

$$\leq \varepsilon \cdot \int_{-\delta}^{\delta} |F_n(y)|\, dy + 2\|f\|_\infty \int_{\delta \leq |y| \leq \pi} |F_n(y)|\, dy, \text{ where } \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

$$\leq \varepsilon \cdot M + 2\|f\|_\infty \varepsilon \quad \text{(by Q1b (2))}$$

$$= \varepsilon (M + 2\|f\|_\infty)$$

$$\therefore \forall \varepsilon > 0,\ \forall n \geq N,\ \left| \int_{-\pi}^{\pi} (f(x-y)-f(x))\, F_n(y)\, dy \right| \leq \varepsilon (M + 2\|f\|_\infty).$$

$$\therefore \lim_{n\to\infty} \int_{-\pi}^{\pi} (f(x-y)-f(x))\, F_n(y)\, dy = 0, \text{ hence } \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, F_n(y)\, dy = f(x).$$

(b) Idea: apply (a) and definition of F_n

Recall that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, F_n(y)\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) \left(\frac{1}{n} \sum_{\ell=0}^{n-1} D_\ell(y) \right) dy$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, D_\ell(y)\, dy = \frac{1}{n} \sum_{\ell=0}^{n-1} S_\ell(f)(x) \quad \text{(See e.g. Tutorial 1, Q3a)}$$

$$= 0 \quad (\because c_k(f) = 0,\ \forall k \in \mathbb{Z})$$

$\therefore$ By (2a), $f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, F_n(y)\, dy = 0.$

Rmk • In Q2a, one observes that the explicit definition of F_n play no role.

In fact, the key ingredients of (2a) are the properties of $\{F_n\}$ as a good kernel.

• Meanwhile, the Dirichlet kernel $\{D_n\}$ is NOT good, so above cannot show

$$S(f)(x) = \lim_{n\to\infty} (f * D_n)(x) = f(x) \text{ for } f \text{ being continous only.}$$

Actually, there exists 2π-periodic continuous function f such that

$S(f)(0)$ diverges, hence $S(f)(0) \neq f(0)$. (See e.g. [Stein: Ch.3, Sec.2.2])

Hence, $\{F_n\}_{n=1}^{\infty}$ is "better" than $\{D_n\}_{n=1}^{\infty}$ in the sense that $(f * F_n)_{n=1}^{\infty}$ has better convergence property than $(f * D_n)_{n=1}^{\infty} = (S_n(f))_{n=1}^{\infty}$.

• Q2b shows the uniqueness property of Fourier series of continuous functions:

If two 2π-periodic continuous functions f, g satisfy $S(f) \equiv S(g)$, then $f \equiv g$.